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CONSTRUCTION OF THE VALUE FUNCTION IN A PURSUIT–EVASION GAME WITH THREE PURSUERS AND ONE EVADER†

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A differential pursuit–evasion game is considered with three pursuers and one evader. It is assumed that all objects (players) have simple motions and that the game takes place in a plane. The control vectors satisfy geometrical constraints and the evader has a superiority in control resources. The game time is fixed. The value functional is the distance between the evader and the nearest pursuer at the end of the game. The problem of determining the value function of the game for any possible position is solved.

Three possible cases for the relative arrangement of the players at an arbitrary time are studied: "one-after-one", "two-after-one", "three-after-one-in-the-middle" and "three-after-one". For each of the relative arrangements of the players a guaranteed result function is constructed. In the first three cases the function is expressed analytically. In the fourth case a piecewise-programmed construction is presented with one switchover, on the basis of which the value of the function is determined numerically. The guaranteed result function is shown to be identical with the game value function. When the initial pursuer positions are fixed in an arbitrary manner there are four game domains depending on their relative positions. The boundary between the "three-after-one-in-the-middle" domain and the "three-after-one" domain is found numerically, and the remaining boundaries are interior Nicomedean conchoids, lines and circles. Programs are written that construct singular manifolds and the value function level lines.

The approach presented in [1–5] is extended. The problem is formalized as in [6, 7] and similar problems have been previously considered in [8–12].

1. THE EQUATIONS OF MOTION AND THE PAYOFF FUNCTIONAL. STATEMENT OF THE PROBLEM

Over the fixed time interval $[t_0, \vartheta]$ we will consider the approach problem for three pursuers $P_i(y_1^{(i)}, y_2^{(i)})$ $(i = 1, 2, 3)$ of the same type and a single vader $E(z_1, z_2)$ in a plane.

The dynamics of the pursuers and evader is given by the equations

$$\dot{y}_1^{(i)} = u_1^{(i)},\ \ \dot{y}_2^{(i)} = u_2^{(i)}\ \ ((u_1^{(i)})^2 + (u_2^{(i)})^2)^{1/2} \leq \mu,\ \ i = 1,2,3 \tag{1.1}$$

$$\dot{z}_1 = \upsilon_1,\ \ \dot{z}_2 = \upsilon_2\ \ (\upsilon_1^2 + \upsilon_2^2)^{1/2} \leq \nu,\ \ \nu > \mu \tag{1.2}$$

where $u^{(i)}$, υ are two-dimensional control vectors.

The payoff functional (PF) σ is the distance between the evader and the pursuer that is nearest to it at the time ϑ

$$\sigma = \min_{i=1,2,3}((z_1(\vartheta) - y_1^{(i)}(\vartheta))^2 + (z_2(\vartheta) - y_2^{(i)}(\vartheta))^2)^{1/2} \tag{1.3}$$

The problem is formalized as in [6, 7].

The pursuers try to minimize, and the evader to maximize the PF. It is required to construct an algorithm for calculating the value function of the game (1.1)–(1.3) for any possible initial position of the game.

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2. TYPICAL RELATIVE POSITIONS

We can identify four typical basic cases for the relative position (Fig. 1): the game of "one-after-one" in which the only significant interaction is between one of the pursuers and the evader P_1 and E_1; the game of "two-after-one" where there is significant interaction between two pursuers and the evader (P_1, P_2 and E_2); the game of "three-after-the-one-in-the-middle" and the game of "three-after-one" in which one must take into account the interaction of all the players (P_1, P_2, P_3, E_0 and P_1, P_2, P_3, E_3, respectively).

3. FEATURES OF THE PROBLEM. AN ALGORITHM FOR DETERMINING THE VALUE OF THE GUARANTEED RESULT FUNCTION IN THE MOST CHARACTERISTIC CASE

We consider the game (1.1)–(1.3) starting at time $t = t_0$ from the typical initial position shown in Fig. 2. We place the origin of Cartesian coordinates at the point $O(t) = (o_1(t), o_2(t))$ equidistant from P_1, P_2 and P_3. We direct the q_2 axis along the perpendicular bisector of the line section $[P_3, P_1]$, and the q_1 axis perpendicular to the q_2 axis. Player E is in the triangle formed by P_1, P_2 and P_3.

The domains of accessibility $G_i(t) = G_i(\vartheta, t, y^{(i)}, (t))$ of the P_i are the circles of radius $r(t) = \mu(\vartheta - t)$ with centres at $(y_1^{(i)}(t), y_2^{(i)}(t))$, and the domain of accessibility $G(t) = G(\vartheta, t, z(t))$ of E is the circle of radius $R(t) = \nu(\vartheta, - t)$, with centre at $(z_1(t), z_2(t))$.

Suppose that the domain of accessibility of the evader E at time t is intersected by the perpendicular bisectors of the line sections P_3P_1, P_1P_2 and P_2P_3 at the points $A_i(t) = (a_1^{(i)}(t), a_2^{(i)}(t))$ $(i = 1, 2, 3)$, respectively. The points $A_i(t)$ $(i = 1, 2, 3)$ and $O(t)$ are called sighting points.

We denote P_i, E, A_i, O at time t_0 by P_{i0}, E_0, A_{i0}, O_0, at time t_* by P_{i*}, E_*, A_{i*}, O_*, at time t_{**} by P_{i**}, E_{**}, A_{i**}, O_{**} and at time τ by $P_{i\tau}$, E_τ, A_τ, O_τ $(i = 1, 2, 3)$.

The pursuers P_i $(i = 1, 2, 3)$ and evader E at times $t = t_0$ are at positions P_{i0} and E_0.

We shall assume that at time $t = t_0$ (Fig. 2) the inequality

$$d(P_{10}, A_{10}) > d(P_{10}, O_0) \tag{3.1}$$

is satisfied where $d(A, B)$ is the Euclidean distance between points A and B.

To investigate the features of problem (1.1)–(1.3) we investigate the following special case. Suppose that throughout the game time interval $[t_0, \vartheta]$ player E chooses the extremal control programme

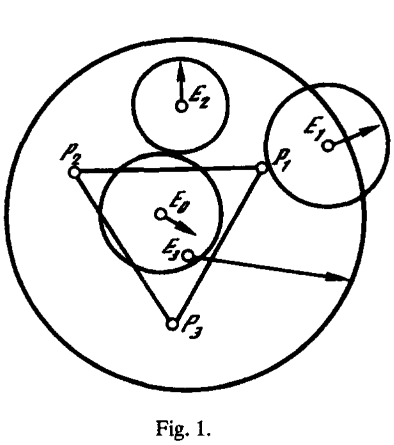


Fig. 1.

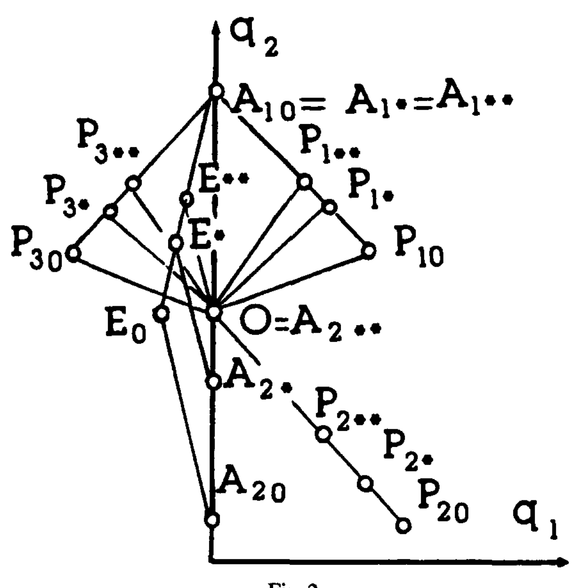


Fig. 2.

$$\upsilon(t) = (\nu\cos\beta^*,\ \nu\sin\beta^*)$$

$$\beta^* = \pi - \mathrm{arctg}[(a_2^1(t_0) - z_2(t_0)) / (a_1^1(t_0) - z_1(t_0))]$$

directed into A_{10} and reports this to the pursuers. (The angles are measured from the positive direction of the q_1 axis.)

In response to this players P_1 and P_2 act as follows. Player P_1 also chooses an extremal control programme

$$u(t) = (\mu\cos\alpha^*,\ \mu\sin\alpha^*)$$

$$\alpha^* = \pi - \mathrm{arctg}[(a_2^1(t_0) - y_2^{(1)}(t_0)) / (a_1^1(t_0) - y_1^{(1)}(t_0))]$$

directed at A_{10} throughout the interval $[t_0, \vartheta]$, player P_2 is controlled arbitrarily subject to the restrictions (1.1), and P_3 moves symmetrically with respect to P_1 relative to the q_2 axis. (It can be shown that the behaviour of P_2 does not affect the value of the PF when the controls of E and P_1, P_3 are as given.) As a result, at some time $t = t_*$, where $t_0 < t_*, < \vartheta$, the players are at positions P_{i*} $(i = 1, 2, 3)$ and E_* (Fig. 2), characterized by the equality

$$d(P_{1*}, A_{10}) = d(P_{1*}, O_*) \tag{3.2}$$

Suppose that the players E, P_1 and P_3 continue their extremal motion towards A_{10} when $t > t_*$, and that player P_2 moves arbitrarily as before. Then when $t > t_*$ the inequality

$$d(P_1(t), A_{10}) < d(P_1(t), O(t)) \tag{3.3}$$

is satisfied, and the difference $\rho(t) = d(P_1(t), O(t)) - d(P_1(t), A_{10})$ will increase monotonically as t increases. Inequality (3.3) and the monotonic growth of $\rho(t)$ hold true right up to time $t = t_{**}$ (Fig. 2) when the inequality

$$O_{**} = A_{2**} \tag{3.4}$$

is satisfied.

At time $t = t_{**}$ a situation arises when player E can ensure himself a larger value of the PF by changing the previous control to a programmed extremal control in the semi-interval $[t_{**}, \vartheta]$ directed at the point O_{**}. As a result, E guarantees himself the value

$$\sigma_1(\vartheta) = d(P_{1**}, A_{10}) - r(t_{**}) \tag{3.5}$$

satisfying the inequality

$$\sigma_1(\vartheta) > \sigma(\vartheta) = d(P_{1**}, A_{10}) - r(t_{**}) \tag{3.6}$$

following from (3.3) and (3.4).

It can be shown that for any control $u_2(t)$ $(t_0 \leqslant t \leqslant t_{**})$ satisfying restriction (1.1) the inequality

$$\sigma(\vartheta) = d(P_{2**}, O_{**}) - r(t_{**}) \geqslant \sigma_1(\vartheta) \tag{3.7}$$

is satisfied.

On the basis of this discussion of the developing situation in this particular case of the game (1.1)–(1.3) one arrives at the following conclusions.

If player E applies the extremal control directed at the point A_1, then it is inadvisable for players P_1, P_3 to use the extremal sighting at the point A_{10} (unlike the case of "two-against-one" [8]).

We now fix any of the game positions that appear in the above case at some time t where $t_* < t \leqslant t_{**}$. We shall take this to be the initial position for some new game (1.1)–(1.3). As before, suppose that in this new game, player E uses an extremal control programme $\upsilon(t)$ directed at the point A_{10}. It can be shown that for any admissible controls for players P_1, P_2 and P_3, the value of the PF $\sigma(\vartheta)$ in the interval $t_* \leqslant t \leqslant t_{**}$ will increase (noting that here the values of t_* and t_{**} will in general differ from their previous values).

The fact that it is impossible to restrain the increase in the value of the PF in certain positions of the game is one singular feature of the problem under consideration. The problem then arises: how does one construct a value function for such game positions? The answer to this question is the main content of this paper.

An algorithm for determining the values of the guaranteed result function (GRF) for the game positions considered above is based on the following considerations. Suppose that the evader E moves extremally towards the point A_1 during the interval $[t_0\vartheta]$. Using the arguments given above we assume that pursuer P_1 chooses an extremal control $u_1(t)$ $(t_0 \leqslant t \leqslant \tau)$ directed at some angle α to the q_1 axis, pursuer P_3 chooses a control $u_2(t)$ directed at an angle $\pi - \alpha$ to q_1, and pursuer P_2 moves extremally towards some point $O(\tau) = A_2(\tau)$ given by the equations

$$d(P_{20}, O_\tau) - \mu\tau = d(P_{1\tau}, A_{10}) = d(P_{3\tau}, A_{10}) \tag{3.8}$$

(see Fig. 3).

The values of τ and α are given by the following conditions.

1. P_1, P_3 and E move extremally, and at time $t = \tau$ should be at positions $P_{1\tau}$, $P_{3\tau}$ and E_τ whose ordinates coincide.

2. P_2, moving extremally towards some point $A_{2\tau}$, should at time $t = \tau$ be at position $P_2(2\tau)$ for which

$$d(P_{20}, A_{2\tau}) - \mu\tau = d(P_{2\tau}, A_{2\tau}) = d(P_{1\tau}, A_{10}) = d(P_{3\tau}, A_{10}) \tag{3.9}$$

holds.

3. The equation

$$O_\tau = A_{2\tau} \tag{3.10}$$

must hold.

Using conditions (3.8)–(3.10) we write out below the equations that determine the specific values of α and $\tau(\alpha)$ for any initial position. Looking ahead, it is necessary, unfortunately, to note that the calculation of α and $\tau(\alpha)$ leads to the need to solve transcendental equations of high degree and that this has to be done numerically.

As a result the required GRF denoted by γ is found from the expressions

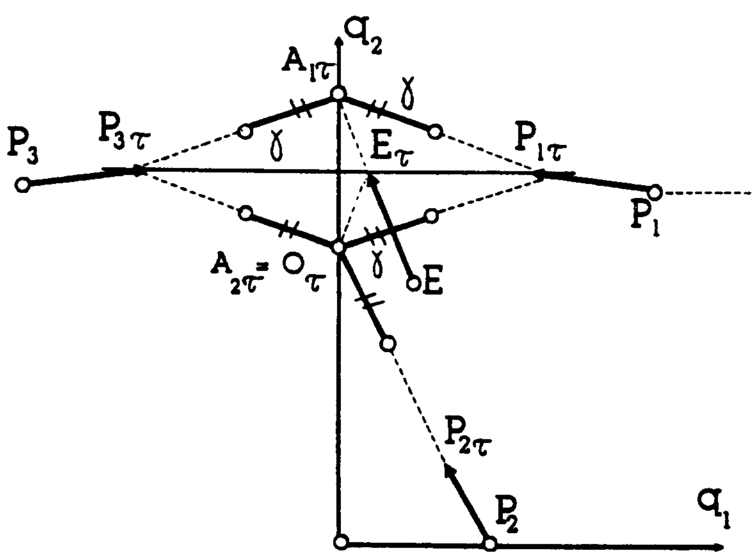


Fig. 3.

$$\gamma = d(P_{1\tau}, A_{10}) - r(\tau) = d(P_{1\tau}, A_{2\tau}) - r(\tau) = d(P_{3\tau}, A_{10}) - r(\tau) =$$
$$= d(P_{3\tau}, A_{2\tau}) - r(\tau) = d(P_{2\tau}, A_{2\tau}) - r(\tau) \quad (3.11)$$

The function γ is υ-stable and this follows directly from the definition of υ-stability by virtue of the linearity of system (1.1)–(1.2) [6]. GRFs corresponding to extremal motion of E towards the points A_{20} and A_{30} are defined similarly.

4. GRF CONSTRUCTION ALGORITHM

For fixed positions P_j ($i = 1, 2, 3$) the plane R_2 of initial positions of E at time t is, according to the typical cases of Section 2, the union of domains D_1, D_2, D_0 and D_3, where D_1 is the domain of the "one-after-one" game, D_2 is the domain of the "two-after-one" game, D_0 is the domain of the "three-after-one-in-the-middle" game and D_3 is the domain of the "three-after-one" game.

These domains are shown in Fig. 4 for different values of μ and ν for fixed $\vartheta = 1$. In Fig. 4(a) $\mu = 12$, $\nu = 24$, in Fig. 4(b) $\mu = 10$, $\nu = 20$ and in Fig. 4(c) $\mu = 6$, $\nu = 12$. It should be said that in Fig. 4(b) the singular lines pass through the point of intersection of the boundaries of two neighbouring D_3 domains, and in Fig. 4(a) they connect the points P_i ($i = 1, 2, 3$) with some point in the interior of the triangle $P_1P_2P_3$ which is found numerically. The boundaries of the domain D_3 in Fig. 4(b) are also found numerically.

We note an obvious property of the domains D_i: $D_i \cap D_j \neq \phi$, ($i, j = 0, 1, 2, 3, i \neq j$) if $D_i \neq \phi$ and $D_j \neq \phi$.

We determine the *GRP* in all three typical cases of the game (1.1)–(1.3).

For the domains D_0, D_1 and D_2 the GRF can be written in the general form

$$\gamma(t, x) = \max_{z \in G(t)} \min_{i} \min_{y \in G_i(t)} d(z, y), \quad i = 1, 2, 3 \quad (4.1)$$

where x is the position of the game.

Note that the maximum with respect to z is reached at an internal point of the domain of accessibility of player E for the domain D_0, and at the boundary for domains D_1 and D_2.

Expression (4.1) gives the general form of the expression for the GRF at each of the domains of the game (1.1)–(1.3).

A value function for the domains D_1 and D_2 was found in [1, 6].

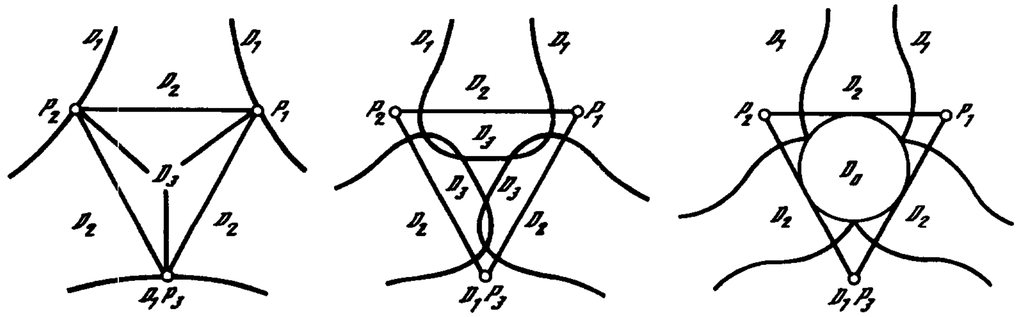
This paper concentrates on the "three-after-one" case considered below.

For each value of the index $j = 1, 2, 3$ we introduce a distinct system of coordinates $q^j_1 q^j_2$.

When $j = 1$ the q^1_1 and q^1_2 axes coincide with those defined in Section 3, when $j = 2, 3$ the coordinate axes $q^j_1 q^j_2$ are obtained from the $q^1_1 q^1_2$ axes by a rotation about the point O, after which the q^j_2 axis coincides with the perpendicular bisectors of the line sections $[P_1P_2]$ and $[P_2P_3]$, respectively.

We will determine the algorithm for constructing the GRFs $\gamma_j(t, x)$ ($j = 1, 2, 3$) corresponding to extremal motion $E(t)$ to $A_j(t)$ in the system of coordinates $q^j_1 q^j_2$.

The extremal motion $E(t)$ and $A_j(t)$ is given by the angle B^*_j ($j = 1, 2, 3$)



(a) (b) (c)

Fig. 4.

$$\beta_j^* = \pi - \mathrm{arctg}[(a_2^j(t_0) - z_2(t)) / (a_1^j(t) - z_1(t))] \tag{4.2}$$

The position of $E(\tau)$ at each instant $\tau > t$ is governed by the expression

$$E(\tau) = (z_1(\tau),\ z_2(\tau)) = (z_1(t) + \nu\tau\cos(\beta_j^*),\ \ z_2(t) + \nu\tau\sin(\beta_j^*)) \tag{4.3}$$

$P_j(t)$ moves extremally towards $A_j(t)$ ($j = 1, 2, 3$) at an angle

$$\alpha_j^* = \pi - \mathrm{arctg}[(a_1^j(t) - y_2^{(j)}(t)) / (a_1^j(t) - y_1^{(j)}(t))] \tag{4.4}$$

to the abscissa axis q_1^j.

We determine the position of P_j moving at an arbitrary angle α to q_1^j at time $\tau > t$ to be

$$P_j(\tau(\alpha)) = (y_1^{(j)}(\tau(\alpha)),\ y_2^{(j)}(\tau(\alpha))) = (y_1^{(j)}(t) + \mu\tau\cos\alpha,$$
$$y_2^{(j)}(t) + \mu\tau\sin\alpha),\quad j = 1,2,3 \tag{4.5}$$

In order to satisfy conditions (3.8)–(3.10) and determine the GRFs γ_j ($j = 1, 2, 3$) from (3.11), the following algorithm is presented for calculating the values of the angle α_j and the time when the ordinates of P_j and E τ_j coincide.

The angle α_j for $j = 1, 2, 3$ and the corresponding τ_j are related by the equation

$$\tau_j(\alpha_j) = (y_2^{(j)}(t) - z_2(t)) / (\nu\sin\beta_j^* - \mu\sin\alpha_j) \tag{4.6}$$

We determine the function $f_j(\alpha_j)$ as follows:

$$f_j(\tau_j(\alpha_j)) = y_2^{(j)}(\tau_j(\alpha_j)) - [a_2^{(j)}(t) + o_2(\tau_j(\alpha_j))] / 2 \tag{4.7}$$

$$o_2^j(\alpha_j) = y_2^{(k)}(t) + [(R_j(\tau_j(\alpha_j)) + \mu\tau(\alpha_j))^2 - (y_1^{(k)}(t))^2]^{1/2} \tag{4.8}$$

where $o_2(\tau_j(\alpha_j))$ is the ordinate of the point $O(\tau_j(\alpha_j))$, $k = 2, 1, 3$ for $i = 1, 2, 3$ and $R_j(\tau_j(\alpha_j))$ is equal to either the distance between P_j and A_j if $y_1^{(j)} \geqslant z_1$, or between E and A_j if $y_1^{(j)} < z_1$, at time $\tau_j(\alpha_j)$

$$R_j(\tau_j(\alpha_j)) = \begin{cases} d(P_j(\tau_j(\alpha_j)),\ A_j(t)), & \text{if}\quad y_1^{(j)}(\tau_j(\alpha_j)) \geqslant z_1(\tau_j(\alpha_j)) \\ d(E(\tau_j(\alpha_j)),\ A_j(t)), & \text{if}\quad y_1^{(j)}(\tau_j(\alpha_j)) < z_1(\tau_j(\alpha_j)) \end{cases} \tag{4.9}$$

Using (4.6)–(4.9) one can calculate the angle α_j depending on the sign of the function $f_j(\tau_j(\alpha_j^*))$. If the sign is negative we take the value of α_j to be equal to the extremal angle α_j^*, otherwise we take it to be the root of the equation $f_j(\alpha) = 0$ which is little different from the extremal value

$$\alpha_j = \begin{cases} \alpha_j^*, & \text{when}\quad f_j(\tau_j(\alpha_j^*)) \leqslant 0 \\ \alpha: f_j(\alpha) = 0,\ \ |\ \alpha_j^* - \alpha\ | \Rightarrow \min, & \text{when}\quad f_j(\tau_j(\alpha_j^*)) > 0 \end{cases} \tag{4.10}$$

Thus, using (4.2)–(4.10) we determine the GRFs γ_j ($j = 1, 2, 3$) to be

$$\gamma_j = R_j(\tau(\alpha_j)) - r(\tau_j(\alpha_j)) \tag{4.11}$$

and as a result we represent the GRF $\gamma(t, x)$ in the form

$$\gamma(t, x) = \max_j \gamma_j(t, x) \tag{4.12}$$

Expressions (4.12) and (4.1) are identical when the inequality $y_1^{(j)}(\tau_j(\alpha_j)) \geqslant z(\tau_j(\alpha_j))$ is satisfied. We note that in the interval $[t, \tau]$ we only have the "three-after-one" case, and at time τ the "three-after-one" case, and at time τ the "three-after-one" case may be identical with any of the remaining game cases.

Thus the GRF $\gamma(t, x)$ is determined as follows:

$$\gamma(t, x) = \max_j \max_{z \in G(\tau_j)} \min_i \min_{y \in G_i(\gamma_j)} d(z, y),\quad j = 1,2,3,\ \ i = 1,2,3 \tag{4.13}$$

where $\tau_j = \tau_j(t, x)$. Expression (4.13) enables us to determine all the boundaries of the domains D_j ($j = 0, 1, 2, 3$).

The curve separating the domains D_1 and D_2 is given by the internal Nicomedean conchoids of radii $R(t)$ with centres at P_j ($j = 1, 2$), and a circle of radius $R(t)$ with centre at O separates domains D_0, D_1 and D_0, D_2, straight lines joining the positions of the pursuers separate D_2 from D_3, and the curve separating domains D_3 and D_0 is found numerically from the condition that the equality

$$\alpha_j = \alpha_0, \quad j = 1, 2, 3 \tag{4.14}$$

is satisfied if E lies on the boundary, with α_j being determined from (4.10) and α_0 being the angle between the ray P_jO and the abscissa.

Singular manifolds of dimensions 1 and 0 are found numerically from the conditions

$$\gamma_i(t) = \gamma_j(t), \quad i \neq j, \quad i, j = 1, 2, 3; \quad \gamma_1(t) = \gamma_2(t) = \gamma_3(t) \tag{4.15}$$

respectively.

5. THE u-STABILITY PROPERTY OF THE GRF

The u-stability property of the GRF in domains D_1 and D_2 was proved in [1, 6]. It remains to verify u-stability in domains D_0 and D_3.

Assertion 1 (u-stability in D_0). Suppose that when $t = t_0$, $x = x_0$ we have $\gamma(t_0, x_0) = \gamma_0$. Then for any position $E \in D_0$ and any constant control $\upsilon = (\upsilon_1, \upsilon_2) = \text{const}$ in the interval $[t_0, t]$ there are controls $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ such that the inequality $\gamma(t, x(t)) = \gamma_1 \leqslant \gamma_0$ holds.

It can be shown that with such controls for the pursuers P_i, $i = 1, 2, 3$ there are controls directed towards $O(t_0)$ which have the form

$$u_j^{(i)} = \mu(o_j(t_0) - y_j^{(i)}(t_0)) / d(P_i(t_0), \ O(t_0)), \quad i = 1, 2, 3, \quad j = 1, 2$$

Remark. If a time $t^* \in [t_0, t]$ exists such that $E(t)^* \in D_i$, $i = 1, 2$, then the chosen controls can be replaced in the interval $[t^*, t]$ by controls corresponding to the cases D_i ($i = 1, 2$).

Assertion 2 (u-stability in D_3, the regular case). Suppose that when $t = t_0$, $x = x_0$ the value of the GRF (4.12) is given by the equality $\gamma(t_0, x_0) = \gamma_1(t_0, x_0) = \gamma_0$. Then for any position $E \in D_3$, ($E \notin S$) and any constant control $\upsilon = (\upsilon_1, \upsilon_2)$ in the interval $[t_0, t]$ one can find controls $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ such that the value of the GRF at time t satisfies the inequality $\gamma(t, x(t)) = \gamma_1 < \gamma_0$.

We choose the controls

$$u_j^{(1)} = \mu(\omega_j - y_j^{(1)}(t_0)) / d(P_1(t_0), \Omega), \quad i = 1, 2, \quad j = 1, 2$$

$$\Omega \in (\omega_1, \omega_2) = B_{R_1}(A_1(t)) \cap B_{\mu\tau_1})P_1(t_0)) \neq \phi, \quad d(\Omega, O(\tau_1)) \Rightarrow \min$$

$$u_1^{(2)} = -u_1^{(1)}, \quad u_2^{(2)} = u_2^{(1)}$$

$$u_j^{(3)} = \mu(o_j(\tau_1) - y_j^{(3)}(t_0)) / d(P_3(t_0), \ O(\tau_1)), \quad j = 1, 2$$

Here $B_R(A)$ is a circle of radius R with centre at A, and the point $\Omega = (\omega_1, \omega_2)$ is given by

$$\Omega = B_{R(t)}(A_1(t)) \cap B_{\mu\tau}(P_1(t_0)) \neq \phi, \quad d(\Omega, O(t_0)) \Rightarrow \min$$

One can verify that these controls along the interval $[t_0, t]$ guarantee that the inequality $\gamma(t, x(t)) < \gamma_0$ is satisfied. Controls for the case when $\gamma(t_0, x_0) = \gamma_j(t_0, x_0) = \gamma_0$, ($j = 2, 3$) are chosen similarly.

Assertion 3 (u-stability in D_3, the singular case). Suppose that when $t = t_0$, $x = x_0$ the equality $\gamma(t_0, x_0) = \gamma_1(t_0, x_0) = \gamma_3(t_0, x_0) > \gamma_2(t_0, x_0)$ is satisfied and also that $\alpha_1 = a_1^*$, $\alpha_3 = \alpha_3^*$ (see (4.10)). The value of the GRF is given by $\gamma_j(t_0, x_0) = \gamma_0$. Then for any position $E(x) \in D_3(x \in S)$ and arbitrary control $(\upsilon_1, \upsilon_2) = \text{const}$ one can find controls $u^{(i)} = (u_1^{(i)}, u_2^{(i)})$ such that at time t the inequality $\gamma(t, x(t)) < \gamma_0$ is satisfied.

Suppose that $G(t)$ intersects the line $A_1(t_0)A_3(t_0)$ at points $A_1(t)$ and $A_3(t)$. We denote the perpendicular bisector of the line section $A_1(t)A_3(t)$ by L and the intersection if L with the circle $B_{\mu(t-t_0)}(P_1(t_0))$ by I.

The assertion demonstrates the following choice of controls: player P_i ($i = 1, 2, 3$) chooses the control $u_j^{(i)}$ directed towards the point $\Omega_i = (\omega_1^{(i)}, \omega_2^{(i)})$

$$u_j^{(i)} = \mu(\omega_j^{(i)} - y_j^{(i)}(t_0)) / d(P_i(t_0), \Omega_i), \quad j = 1, 2$$

$$\Omega_1 = \begin{cases} \Omega: \ \Omega \in I, \ I \neq \phi \\ A_i(t): d(A_i(t)P_1(t_0)) > d(A_j(t)P_1(t_0)), \ i, j = 1, 2, \ I = \phi \end{cases}$$

After the point Ω_1 we find $P_1(t)$ and the value $V = d(P_1(t), A_1(t))$, while the points Ω_2 and Ω_3, each of which belongs to the intersection of two circles, are situated maximally close to one another

$$\Omega_2 \in \{B_{\mu(t-t_0)}(P_2(t_0)) \cap B_V(A_1(t))\}$$

$$\Omega_3 \in \{B_{\mu(t-t_0)}(P_3(t_0)) \cap B_V(A_3(t))\}$$

$$d(\Omega_2, \Omega_3) \Rightarrow \min$$

One can similarly prove the assertion with the assumption that the maximum of the GRF (4.12) is reached on γ_2 and γ_3 or on γ_1 and γ_2. The remark for Assertion 1 is also true for Assertions 2 and 3.

According to the data of the numerical investigation the singular lines are straight lines.

In conclusion, we will consider the case when the singular manifolds are given by the second relation of (4.10) where $\alpha_i \neq \alpha_i^*$ (= 1, 2, 3).

Because the transcendental equation $f_j(\alpha) = 0, j = 1, 2, 3$ is of greater than fourth degree in sin (α) the calculation of the GRF is extremely difficulty without using a computer.

We proceed as follows. Let s_{ij} be the nodes of an orthogonal grid defend in the domain D_3. We denote the rectangle formed by the nodes $s_{i-1,j-1}, s_{i-1,j}, s_{ij-1}, s_{ij}$, by S_{ij}. It is obvious that for any $E \in S$ one can find a S_{ij} such that $E \in S_{ij}$.

As above, the evader chooses the control υ = const in the interval $[t_0, t]$.

Proposition. If the u-stability property holds at the nodes defining S_{ij}, then it also holds at any position $E \in S_{ij}$.

The proof of u-stability at the nodes s_{ij} was carried out numerically using a program implementing the GRF construction algorithm (4.2)–(4.13), minimizing the GRF and generating the position $x(t)$ and the value of the GRF for the computed $u^{(i)}$ ($i = 1, 2, 3$) for an arbitrary control of the evader; here the subdivision step was chosen depending on the desired accuracy.

It was verified numerically that the u-stability property holds at the nodes of S_{ij} to any required degree of accuracy.

Figure 5 shows the dependence of the GRF $\gamma(t, x)$ on different controls υ for the position determined by the second of conditions (4.10). It is clear that $\gamma(t, x) \leqslant \gamma(t_0, x_0)$ with equality only holding for extremal controls υ determined from (4.3).

Figure 6 shows the level lines of the game value function in all the D_i ($i = 0, 1, 2, 3$) for fixed positions of the pursuers $P_1(100/3^{1/2}, 0)$, $P_2(-100/3^{1/2}, 0)$, $P_3(0, -100)$ when $\vartheta = 1$. In Fig. 6(a) $\mu = 12$, $\nu = 24$, in

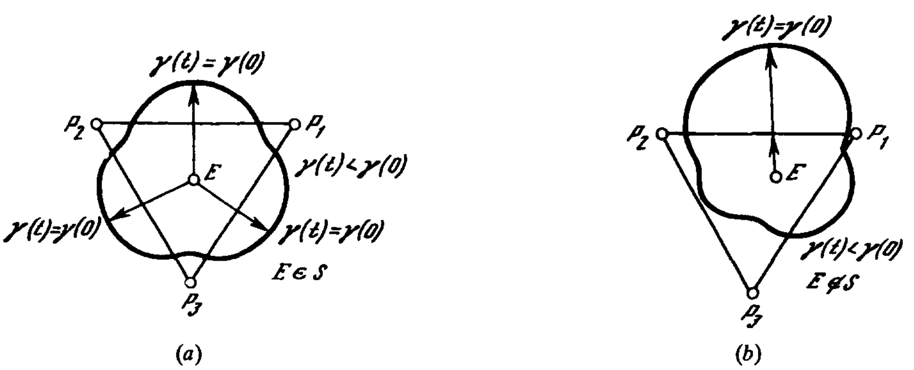
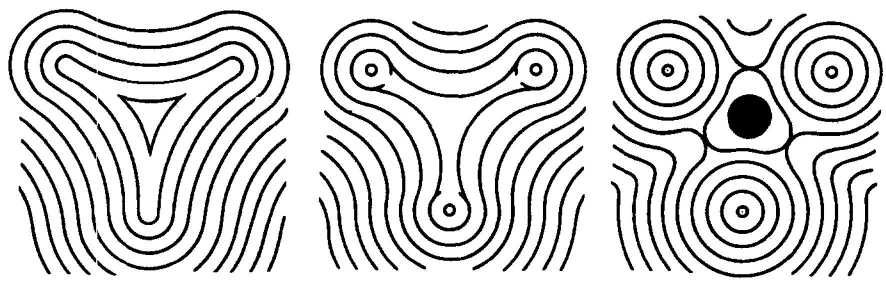


Fig. 5.



(a) (b) (c)

Fig. 6.

Fig. 6(b) $\mu = 10, \nu = 20$ and in Fig. 6(c) $\mu = 6, \nu = 12$. The level lines in Fig. 6 correspond to domains of the initial positions of the evader in Fig. 4.

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